JOURNAL OF APPROXIMATION THEORY 77, 202–211 (1994)

# Comparing Two Versions of Markov's Inequality on Compact Sets

JOHAN LITHNER

*Department of Mathematics,*
*University of Umeå, S-901 87 Umeå, Sweden*

*Communicated by Vilmos Totik*

Received August 27, 1992; accepted in revised form October 27, 1993

We compare a local and a global version of Markov's inequality defined on compact subsets of $\mathbb{C}$. As a main result we show that the local version implies the global one. The same result was also obtained independently by A. Volberg. © 1994 Academic Press, Inc.

## NOTATION

The following basic notation will be used throughout the paper. $\mathscr{P}_n(\mathbb{C})$ denotes the set of algebraic polynomials from $\mathbb{C}$ of degree at most $n$. $B(z, r) \subset \mathbb{C}$ is the closed disc with center $z$ and radius $r$. If $A, B \subset \mathbb{C}$ then the supremum norm of a function $f$ over $A$ is denoted by $\|f\|_A$, the transfinite diameter of $A$ by $d(A)$ and $\operatorname{dist}(A, B)$ is the distance between the sets $A$ and $B$. $\hat{\mathbb{C}}$ is the extended complex plane.

## 1. INTRODUCTION

A. A. Markov's well-known inequality from 1889 states that

$$\|P'\|_{[-1,1]} \leqslant n^2 \|P\|_{[-1,1]}, \qquad \text{for all} \quad P \in \mathscr{P}_n(\mathbb{R}).$$

There are many ways of modifying this theorem to get conditions on more general subsets of $\mathbb{C}$, and we shall study the following two versions.

DEFINITION 1.1. A compact non-empty subset $E$ of $\mathbb{C}$ is said to preserve *the global Markov inequality* (GMI) if there exist constants $M > 0$ and $r > 0$ depending only on $E$, such that for every $n \geqslant 1$

$$\|P'\|_E \leqslant Mn^r \|P\|_E, \qquad \text{for all} \quad P \in \mathscr{P}_n(\mathbb{C}). \qquad \text{(GMI)}$$



0021-9045/94 $6.00

This version of Markov's inequality is applied, for example, in connection with extension and approximation of $C^\infty$ functions (see. e.g., [8]) and Bernstein-type theorems (see, e.g., [5]).

If $E$ is a compact subset of $\mathbb{C}$, let $g(z, w)$ be the Green function of the unbounded component of $\hat{\mathbb{C}}\setminus E$ with pole at $w$.

DEFINITION 1.2. For a compact subset $E$ of $\mathbb{C}$, the Green function has *the Hölder continuity property* (HCP) if there exist constants $M>0$ and $r>0$ depending only on $E$, such that

$$g(z, \infty)\leqslant M\delta^r, \qquad \text{if} \quad \operatorname{dist}(z, E)\leqslant\delta\leqslant 1. \tag{HCP}$$

If $E$ is a (HCP) set, then by the Bernstein–Walsh lemma [11, p. 77]

$$|P(z)|\leqslant \exp M\ \|P\|_E, \qquad \text{for all} \quad P\in\mathscr{P}_n(\mathbb{C}) \text{ if } \operatorname{dist}(z, E)\leqslant 1/n^r.$$

Applying Cauchy's integral formula one can easily show that this implies that $E$ preserves the global Markov inequality.

DEFINITION 1.3. A closed subset $F$ of $\mathbb{C}$ is said to preserve *the local Markov inequality* (LMI) if for every $n\geqslant 1$ there exists a constant $c=c(F, n)>0$ such that for all polynomials $P\in\mathscr{P}_n(\mathbb{C})$ and all $\delta\in(0, 1]$ the inequality

$$\|P'\|_{B\cap F}\leqslant\frac{c}{\delta}\|P\|_{B\cap F} \tag{LMI}$$

is fulfilled, where $B$ is any closed disc with radius of length $\delta$, centered at $F$.

One example where the local Markov inequality has been used is [4], in extension theorems of Whitney-type for function spaces on compact subsets on $\mathbb{R}^N$.

Then how are these two versions of Markov's inequality related? It is shown in [7] that certain subsets of $\mathbb{R}^N$ with polynomial cusps satisfy (GMI). For $N\geqslant 2$ one can immediately conclude from [4, Thm. 2, p. 38] that these sets do not preserve the local Markov inequality.

In [1] it has recently been proved that the ordinary Cantor ternary set has the Hölder continuity property. In Section 2 we construct a family $\mathscr{E}$ of sets that are generalizations of the Cantor ternary set, and in Sections 3 and 4 we extend the method of [1] to show the (HCP) property uniformly for all sets in $\mathscr{E}$. It is known (see [10, proof of Thm. 2], slightly adjusted for subsets of $\mathbb{C}^N$ instead of $\mathbb{R}^N$) that if a set $F$ preserves the local Markov inequality then $F$ can be regarded as a union of Cantor type sets. In Section 5 we show that if $F\subset\mathbb{C}$ then these sets belong essentially to $\mathscr{E}$, so we finally conclude that $F$ also preserves the global Markov inequality.

## 2. A Family of Cantor Type Sets

We shall now describe the construction of a generalized Cantor type subset of $\mathbb{C}$. From now on, let $0 < q \leqslant \frac{1}{3}$ be a fixed constant and set $B_{0,1} := B(0, \frac{1}{2})$. If $B_{k,n}$ is constructed, then choose two new subdiscs $B_{k+1,l}$ and $B_{k+1,l+1}$ of $B_{k,n}$ such that

(i) $\text{radius}(B_{k+1,l}) = \text{radius}(B_{k+1,l+1}) = q^{k+1}/2$

(ii) $\text{dist}(B_{k+1,l}, B_{k+1,l+1}) \geqslant q^{k+1}$.

Let $E$ be the Cantor type set defined by

$$E := \bigcap_{k=0}^{\infty} \bigcup_{n=1}^{2^k} B_{k,n}.$$

For our calculations later on we need to attach a neighborhood to each $B_{k,n}$ in the following way. Let $Q_{k,n}$ be a closed disc with the same center as $B_{k,n}$. For $k \geqslant 1$, if $B_{k,n}$ and $B_{k,n+1}$ are subsets of the same disc $B_{k-1,m}$ choose the radius of $Q_{k,n}$ to be half the distance between the centers of $B_{k,n}$ and $B_{k,n+1}$ (see Fig. 1), that is, somewhere in the interval $[q^k, \frac{1}{2}(1-q)\, q^{k-1}]$. Finally, choose $\text{radius}(Q_{0,1})$ to be any number in the interval $[1, (1-q)/2q]$.

The ordinary Cantor ternary set is self-similar in the sense that locally it looks like a smaller version of the whole set. We are going to use the fact that locally $E$ may not look like a smaller version of itself, but like a smaller version of another set constructed as above, possibly with other choices in step (ii) and in the choice of $\text{radius}(Q_{0,1})$. More precisely, take $B_{k,n} \cap E$, translate it so that $B_{k,n}$ is centered at 0 and enlarge this set by a ratio $q^{-k}$. This new set could have been constructed exactly as described above and since we are going to use this kind of linear transformations it is essential that we study the whole class of Cantor type sets simultaneously:

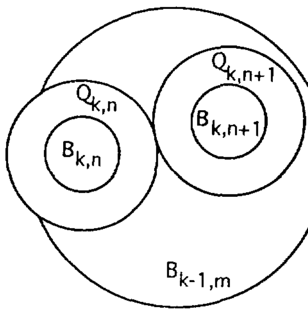


Figure 1

For $q$ still fixed let $\mathscr{E} := \{E^i\}_{i \in I}$ be the family of all possible Cantor type sets constructed with $B^i_{k,n}$ as above together with corresponding $Q^i_{k,n}$, where $I$ is some index set.

## 3. The Green Function and the Harmonic Measure

Any $E^i \in \mathscr{E}$ is a regular set of the Dirichlet problem by Wiener's criterion [9, Thm. III.64] so we can define $\omega^i_{k,n}(z)$, *the harmonic measure of* $B^i_{k,n} \cap E^i$ *with respect to the complement of* $E^i$ to be the unique function harmonic on $\hat{\mathbb{C}} \backslash E^i$ and continuous on $\hat{\mathbb{C}}$ such that $\omega^i_{k,n} = 1$ on $B^i_{k,n} \cap E^i$ and $\omega^i_{k,n} = 0$ o $E^i \backslash (B^i_{k,n} \cap E^i)$. If $B^i_{k+1,l} \cup B^i_{k+1,l+1} \subset B^i_{k,n}$ then it is easy to see that

$$\omega^i_{k+1,l}(z) + \omega^i_{k+1,l+1}(z) = \omega^i_{k,n}(z) \qquad \text{for all} \quad z \in \hat{\mathbb{C}}. \tag{3.1}$$

Let $b^i_{k,n}$ be the center of $B^i_{k,n}$ and let $\gamma^i_{k,n}$ be the circle with center $b^i_{k,n}$ and radius $\frac{5}{6}q^k$. If $B^i_{k+1,l} \cup B^i_{k+1,l+1} \subset B^i_{k,n}$ then let $\lambda^i_{k,n}$ be a point on the circle with center $b^i_{k,n}$ and radius $\frac{2}{3}q^k$ such that $\lambda^i_{k,n}$ lies outside $Q^i_{k+1,l} \cup Q^i_{k+1,l+1}$ (see Fig. 2).

Before stating Proposition 3.4, we need some preliminary lemmas.

Lemma 3.1. *There exists a constant $b_1 > 0$ independent of $i$ such that for all $k$, $n$*

$$b_1 \leqslant q^{-k} \, d(B^i_{k,n} \cap E^i).$$

*Proof.* Since $\mathscr{E}$ consists of all Cantor type sets constructed as in Section 3, there exists a $j \in I$ such that $(B^i_{k,n} \cap E^i)$ is the image of the linear transformation $u^i_{k,n}(z) = q^k z + b^i_{k,n}$ from $E^j$. Then we get [9, Thm. III.4]

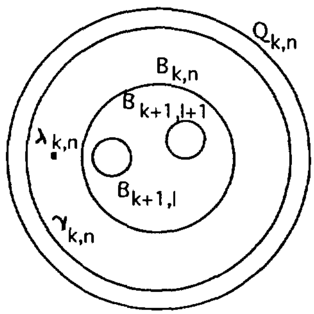


Figure 2

that $d(B^i_{k,n} \cap E^i) = q^k d(E^j)$. In [9, Thm. III.64] it is proved by using Wiener's criterion that $d(E^j) \geq q^2$. Set $b_1 := q^2$. ∎

LEMMA 3.2. *There exists a constant $b_2 > 0$ independent of $i$ such that for all $k, n$*

$$b_2 \leq g^i(z, \lambda^i_{k,n}) \qquad \textit{for all} \quad z \in \gamma^i_{k,n},$$

*where $g^i(z, w)$ is the Green function of $\hat{\mathbb{C}} \setminus E^i$ with pole at $w$.*

*Proof.* Let $\tilde{g}^i_{k,n}(z)$ be the Green function for $\operatorname{int}(Q^i_{k,n} \setminus E^i)$ with pole at $\lambda^i_{k,n}$ and let $\tilde{G}^i(z)$ be the Green function for $\operatorname{int}(Q^i_{0,1} \setminus B(0, \frac{1}{2}))$ with pole at $\lambda^i_{0,1}$. Since radius$(Q^i_{0,1}) \geq 1$ there exists a constant $b_2 > 0$ independent of $i$ such that

$$b_2 \leq \tilde{G}^i(z) \underset{\text{(Max. Principle)}}{\leq} \tilde{g}^i_{0,1}(z) \qquad \text{for all} \quad z \in \gamma^i_{0,1}.$$

$\tilde{g}^i_{k,n}(z)$, $\lambda^i_{k,n}$, and $\gamma^i_{k,n}$ can be reproduced from $\tilde{g}^j_{0,1}(z)$, $\lambda^j_{0,1}$, and $\gamma^j_{0,1}$ for some $j \in I$ by the same linear transformation as in the proof of Lemma 3.1. Therefore for all $k, n$

$$b_2 \leq \tilde{g}^i_{k,n}(z) \underset{\text{(Max. Principle)}}{\leq} g^i(z, \lambda^i_{k,n}) \qquad \text{for all} \quad z \in \gamma^i_{k,n}. \quad ∎$$

LEMMA 3.3. *There exists a constant $b_3 > 0$ independent of $i$ such that for all $k, n$*

$$b_3 \leq \omega^i_{k,n}(z) < 1 \qquad \textit{for all} \quad z \in \gamma^i_{k,n}.$$

*Proof.* $\omega^i_{k,n}(z) < 1$ follows from the Maximum Principle.

Let $\tilde{\omega}^i_{k,n}(z)$ be the unique function harmonic on $\operatorname{int}(Q^i_{k,n} \setminus E^i)$ and continuous on $Q^i_{k,n}$ such that $\tilde{\omega}^i_{k,n}(z) = 1$ on $(B^i_{k,n} \cap E^i)$ and $\tilde{\omega}^i_{k,n}(z) = 0$ on the boundary of $Q^i_{k,n}$. Let $j$ be as in the proof of Lemma 1. By [3, p. 138] we can write

$$\tilde{\omega}^j_{0,1}(z) = \int G^j(z, w)\, d\mu_{E^j}(w),$$

where $G^j(z, w)$ is the Green function for $Q^j_{0,1}$ and $\mu_{E^j}$ is a measure supported on $E^j$ such that $\mu_{E^j}(E^j) = d(E^j)$. By Lemma 3.1, $\mu_{E^j}(E^j)$ is bounded below, and it is easy to estimate $G^j(z, w)$ for $z \in \gamma^j_{0,1}$ and $w \in E^j$ to get a constant $b_3$ such that

$$b_3 \leq \tilde{\omega}^j_{0,1}(z) \qquad \text{for all} \quad z \in \gamma^j_{0,1}.$$

Using the same linear transformation as in Lemma 3.1 and the fact that $\tilde{\omega}^i_{k,n} \leq \omega^i_{k,n}$, we get Lemma 3.3. ∎

If $B^i_{k+1,l} \cup B^i_{k+1,l+1} \subset B^i_{k,n}$ then we define

$$A^i_{k,n} := Q^i_{k,n} \backslash (Q^i_{k+1,l} \cup Q^i_{k+1,l+1}).$$

The following proposition is an extension of [6, Lemma 3.4].

PROPOSITION 3.4. *There exist positive constants $c_1, c_2$ independent of $i$ such that for all $k, n$*

$$c_1 \omega^i_{k,n}(\infty) \leqslant g^i(z, \infty) \leqslant c_2 \omega^i_{k,n}(\infty) \qquad \text{for all} \quad z \in A^i_{k,n}.$$

*Proof.* Let $\hat{g}^i_{k,n}(z, w)$ be the Green function for $\hat{\mathbb{C}} \backslash (B^i_{k,n} \cap E^i)$ with a pole at $w$. We can write [9, Thm. III.37]

$$\hat{g}^i_{k,n}(z, \infty) = \int \ln \frac{|z-w|}{d(B^i_{k,n} \cap E^i)} \, d\mu^i_{k,n}(w), \tag{3.2}$$

where $\mu^i_{k,n}$ is the equilibrium distribution measure of $B^i_{k,n} \cap E^i$. Thus $\mu^i_{k,n}$ is supported on $B^i_{k,n} \cap E^i$ and $\mu^i_{k,n}(B^i_{k,n} \cap E^i) = 1$. From the definitions above we have that if $z \in B^i_{k,n} \cap E^i$ then $|\lambda^i_{k,n} - z| \leqslant 7q^k/6$ and by Lemma 3.1 we get the estimate

$$\hat{g}^i_{k,n}(\lambda^i_{k,n}, \infty) \leqslant \ln \frac{7}{6b_1}. \tag{3.3}$$

Now, the image of $\operatorname{int}(B(0, 1))$ by the function

$$h^i_{k,n}(z) := \frac{3q^k}{4z} + b^i_{k,n}$$

is the complement of $B(b^i_{k,n}, 3q^k/4)$. The composition of $h^i_{k,n}$ and $\hat{g}^i_{k,n}(z, \lambda^i_{k,n})$ is a positive harmonic function on $\operatorname{int}(B(0, 1))$. By Harnack's inequality [2, Thm. 1.18]

$$\hat{g}^i_{k,n}(h^i_{k,n}(w), \lambda^i_{k,n}) \leqslant 19 \hat{g}^i_{k,n}(h^i_{k,n}(0), \lambda^i_{k,n}) \qquad \text{for all} \quad w \in B(0, \tfrac{9}{10}).$$

Since $\gamma^i_{k,n}$ is the image of the circle with center 0 and radius $\frac{9}{10}$ under the mapping $h^i_{k,n}$ we get that

$$\hat{g}^i_{k,n}(z, \lambda^i_{k,n}) \leqslant 19 \hat{g}^i_{k,n}(\infty, \lambda^i_{k,n}) \qquad \text{for all} \quad z \in \gamma^i_{k,n}. \tag{3.4}$$

Furthermore, by the symmetry of the Green function $(g(z, w) = g(w, z))$,

$$g^i(z, \lambda^i_{k,n}) \underset{\text{(Max. principle)}}{\leqslant} \hat{g}^i_{k,n}(z, \lambda^i_{k,n}) \underset{(3.3),(3.4)}{\leqslant} 19 \ln \frac{7}{6b_1} \qquad \text{for all} \quad z \in \gamma^i_{k,n},$$

which together with Lemma 3.2 and Lemma 3.3 yields that there exist constants $c_3, c_4 > 0$ independent of $i$ such that

$$c_3 \omega^i_{k,n}(z) \leqslant g^i(z; \lambda^i_{k,n}) \leqslant c_4 \omega^i_{k,n}(z) \qquad \text{for all} \quad z \in \gamma^i_{k,n}.$$

$\omega^i_{k,n}(z)$ and $g^i(z, \lambda^i_{k,n})$ are harmonic on $\hat{\mathbb{C}} \setminus E^i$ and tend to zero as $z$ tends to $E^i$ outside $\gamma^i_{k,n}$. By the maximum principle the inequality above is valid for all $z$ outside $\gamma^i_{k,n}$, in particular, for $z = \infty$:

$$c_3 \omega^i_{k,n}(\infty) \leqslant g^i(\infty, \lambda^i_{k,n}) \leqslant c_4 \omega^i_{k,n}(\infty). \tag{3.5}$$

Finally, let $u^i_{k,n}$ be as in Lemma 3.1 so that $u^i_{k,n}(A^j_{0,1}) = A^i_{k,n}$. Since $A^j_{0,1}$ can be covered by a finite number of discs (the number independent of $j$) disjoint from $E^j$ and $\lambda^j_{0,1} \in A^j_{0,1}$ we see by Harnack's inequality that there exist constants $c_5, c_6 > 0$ independent of $i$ and $j$ such that

$$\begin{aligned} c_5\, g^i(u^i_{k,n}(\lambda^j_{0,1}), \infty) &\leqslant g^i(u^i_{k,n}(\omega), \infty) \\ &\leqslant c_6\, g^i(u^i_{k,n}(\lambda^j_{0,1}), \infty) \qquad \text{for all} \quad w \in A^j_{0,1}, \end{aligned}$$

and since $u^i_{k,n}(\lambda^j_{0,1}) = \lambda^i_{k,n}$ we get that

$$c_5\, g^i(\lambda^i_{k,n}, \infty) \leqslant g^i(z, \infty) \leqslant c_6\, g^i(\lambda^i_{k,n}, \infty) \qquad \text{for all} \quad z \in A^i_{k,n},$$

which together with (3.5) completes the proof of Proposition 3.4. ∎

COROLLARY 3.5. *There exist positive constants $c_7, c_8$ independent of $i$ such that for all $k, n$ if $B^i_{k,n} \cup B^i_{k,n+1} \subset B^i_{k-1,m}$ then*

$$c_7 \omega^i_{k,n}(\infty) \leqslant \omega^i_{k,n+1}(\infty) \leqslant c_8 \omega^i_{k,n}(\infty).$$

*Proof.* Applying Proposition 3.4 to $z \in A^i_{k,n} \cap A^i_{k,n+1} \neq \emptyset$ gives the corollary. ∎

## 4. THE (HCP) PROPERTY OF THE GENERALIZED CANTOR SET

THEOREM 4.1. *Every set $E^i \in \mathscr{E}$ has the* (HCP) *property with constants $M, r > 0$ independent of $i$.*

*Remark.* $M$ and $r$ depend of course on the costant $q$, fixed in the construction of $\mathscr{E}$.

*Proof.* If $z \notin Q^i_{0,1}$ then $\delta > \frac{1}{2}$. Since $\operatorname{dist}(z, E^i) \leqslant 1$ and $E^i \subset B(0, \frac{1}{2})$ we get by Lemma 4.1 and (3.2) that there exists a constant $c_9$ independent of $i$ such that $g^i(z, \infty) \leqslant c_9$. Then we get (HCP) with $M = 2c_9$ and $r = 1$.

If $z \in Q^i_{0,1}$, $z \notin E^i$ then $z$ belongs to one of the $A^i_{k,n}$ and

$$\delta = \operatorname{dist}(z, E^i) \geqslant \operatorname{dist}(A^i_{k,n}, E^i) \geqslant q^{k+1}/2 \Rightarrow k \geqslant \frac{\ln 2\delta}{\ln q} - 1.$$

If $B^i_{k,n} \cup B^i_{k,n+1} \subset B^i_{k-1,m}$ then

$$\omega^i_{k,n}(\infty) + c_7\omega^i_{k,n}(\infty) \underset{\text{(Corollary 3.5)}}{\leqslant} \omega^i_{k,n}(\infty) + \omega^i_{k,n+1}(\infty) \underset{(3.1)}{=} \omega^i_{k-1,m}(\infty)$$

$$\Rightarrow \omega^i_{k,n}(\infty) \leqslant \frac{1}{1+c_7}\,\omega^i_{k-1,m}(\infty) \leqslant \cdots \leqslant \frac{1}{(1+c_7)^k}\,\omega^i_{0,1}(\infty).$$

Proposition 3.4 then implies

$$g^i(z, \infty) \leqslant c_2\omega^i_{k,n}(\infty) \leqslant c_2\frac{1}{(1+c_7)^k}\,\omega^i_{0,1}(\infty)$$

$$\leqslant c_2\frac{1}{(1+c_7)^{(\ln 2\delta/\ln q)-1}} = M\delta^r,$$

with

$$M = 2^r c_2(1+c_7),\ r = -\frac{\ln(1+c_7)}{\ln q}. \quad \blacksquare$$

## 5. The (HCP) Property of Local Markov Sets

Let $F \subset \mathbb{C}$ be a compact set preserving the local Markov inequality. This property is equivalent [4, Prop. 4 p. 37] to the fact that (LMI) holds for all polynomials of degree one. This in turn leads to the following geometric characterization of local Markov sets [10, Prop. 7]:

Proposition 5.1. *$F$ preserves the local Markov inequality if and only if there exists a constant $c_0 \geqslant 1$ such that for all $z \in F$ and $0 < r \leqslant 1$ there exists a point in $B(z, r) \cap F$ at distance larger than or equal to $r/c_0$ from $z$. $c_0$ depends only on the constant $c$ in* (LMI) *for first degree polynomials.*

*Remark.* A set geometrically characterized in this way is sometimes called *uniformly perfect* or *perfect of the class* 1.

The proof of this actually deals with the case $F \subset \mathbb{R}^N$ but can easily be adapted to compact subsets of $\mathbb{C}^N$. In [10, Thm. 2], Proposition 5.1 is used to prove that $F$ has positive Hausdorff dimension. We are going to use a part of their proof, slightly modified, to show that $F$ is a union of Cantor type sets.

Take a point $z \in F$, set $d_{1,1} = z$ and $\rho = 1/(1+4c_0)$. By Proposition 5.1 there exists a point $d_{1,2} \in B(d_{1,1}, 2\rho c_0) \cap F$ at a distance at least $2\rho$ from $d_{1,1}$. Set $D_{1,1} = B(d_{1,1}, \rho/2)$ and $D_{1,2} = B(d_{1,2}, \rho/2)$. Then $D_{1,1}$ and $D_{1,2}$ are subsets of $D_{0,1} := B(d_{1,1}, \frac{1}{2})$ and $\operatorname{dist}(D_{1,1}, D_{1,2}) \geqslant \rho$.

We continue by induction. Suppose $D_{k,n} = B(d_{k,n}, \rho^k/2)$ is constructed. Set $d_{k+1,l} = d_{k,n}$. By Proposition 5.1 there is a point $d_{k+1,l+1} \in B(d_{k+1,l}, 2\rho^{k+1}c_0) \cap F$ such that $D_{k+1,l} := B(d_{k+1,l}, \rho^{k+1}/2)$ and $D_{k+1,l+1} := B(d_{k+1,l+1}, \rho^{k+1}/2)$ are subsets of $D_{k,n}$ and $\operatorname{dist}(D_{k+1,l}, D_{k+1,l+1}) \geqslant \rho^{k+1}$. Set

$$F_z := \bigcap_{k=0}^{\infty} \bigcup_{n=1}^{2^k} D_{k,n}.$$

It follows that $F_z \subset F \cap B(z, \frac{1}{2})$ from the construction and the fact that $F$ is closed. We see now that apart from being a subset of $B(z, \frac{1}{2})$ instead of $B(0, \frac{1}{2})$, $F_z$ is constructed exactly the same way as one of the sets $E^i$ from Section 2 with $q$ replaced by $\rho$. But since the (HCP) property is invariant under translations and $z \in E$ was arbitrarily choosen, we get from Theorem 4.1 that $F_z$ has the (HCP) property with constants $M$ and $r$ independent of $z$. Finally, since

$$F = \bigcup_{z \in F} F_z,$$

$F$ also has the (HCP) property with the same constants $M$ and $r$ and we get our main result:

THEOREM 5.2. *Suppose $F$ is a compact subset of $\mathbb{C}$ preserving the local Markov inequality. Then $F$ has the* (HCP) *property and consequently preserves the global Markov inequality.*

## REFERENCES

1. L. BIAŁAS AND A. VOLBERG, "Markov's Property of the Cantor Ternary Set," *Studia Math.* **104** (1993), 259–268.
2. W. K. HAYMAN AND P. B. KENNEDY, "Subharmonic Functions," Vol. I, Academic Press, New York, 1976.
3. L. L. HELMS, Introduction to potential theory, Wiley–Interscience, New York, 1969.
4. A. JONSSON AND H. WALLIN, "Function Spaces on Subsets of $\mathbb{R}^n$," Harwood, London/Paris/Utrecht/New York, 1984.

5. J. LITHNER AND A. P. WÓJCIK, "A Note on Bernstein's Theorems," No. 3, Department of Math., Univ. of Umeå, 1992.
6. M. MAKAROV AND A. VOLBERG, On the harmonic measure of discontinuous fractals, LOMI preprints, E-6-86, Leningrad, 1986.
7. W. PAWŁUCKI AND W. PLEŚNIAK, Markov's inequality and $C^\infty$ functions on sets with polynomial cusps, *Math. Ann.* **275** (1986), 467–480.
8. W. PLEŚNIAK, Markov's inequality and the existence of an extension operator for $C^\infty$ functions, *J. Approx. Theory* **61** (1990), 106–117.
9. M. TSUJI, "Potential Theory in Modern Function Theory," Maruzen, Tokyo, 1959.
10. H. WALLIN AND P. WINGREN, Dimensions and geometry of sets defined by polynomial inequalities, *J. Approx. Theory* **69** (1992), 231–249.
11. J. L. WALSH, "Interpolation and Approximation by Rational Functions in the Complex Domain," 4th ed., Colloquium Publications, Vol. 20, American Mathematical Society, Providence, RI, 1965.